

# On associative algebras, modules and twisted modules for vertex operator algebras

Jinwei Yang

## Abstract

We give a new construction of functors from the category of modules for the associative algebras  $A_n(V)$  and  $A_g(V)$  associated with a vertex operator algebra  $V$ , defined by Dong, Li and Mason, to the category of admissible  $V$ -modules and admissible twisted  $V$ -modules, respectively, using the method developed in the joint work [HY1] with Y.-Z. Huang. The functors were first constructed by Dong, Li and Mason, but the importance of the new method, as in [HY1], is that we can apply the method to study objects without the commutator formula in the representation theory of vertex operator algebras.

## 1 Introduction

This paper is a continuation of the paper [HY1]. The aim is to prove results in the representation theory of vertex operator algebras, in particular, for modules, without using the standard commutator formula. The commutator formula for vertex operators plays a very important role in the representation theory of vertex operator algebras since it allows one to apply many techniques in Lie algebra representation theory to study vertex operator algebras and their modules. However, for some important objects in the representation theory of vertex operator algebras such as intertwining operators (or more generally, logarithmic intertwining operators), there is no commutator formula for two intertwining operators and therefore we have to use the associativity of intertwining operators.

In [HY1], jointly with Y.-Z. Huang, the author gave a formula for the residues of certain formal series involving iterates of vertex operators obtained using the weak associativity and the lower truncation property of vertex operators. We proved that the weak associativity for an admissible module is equivalent to this residue formula together with a formula that expresses products of components of vertex operators as linear combinations of iterates of components of vertex operators given in [DLM1] and [L]. We applied this result to give a new construction of admissible modules for an  $N$ -graded vertex algebra  $V$  from modules for its Zhu algebra  $A(V)$ .

In this paper, we use the method in [HY1], but in more general settings, to construct a functor from the category of modules for the associative algebra  $A_n(V)$ , defined in [DLM1], generalizing the Zhu algebra associated with a vertex operator algebra  $V$  for  $n \in \mathbb{N}$ , to the category of admissible  $V$ -modules. We also use the method in [HY1] to construct a functor from the category of modules for the “twisted” generalization of the Zhu algebra  $A_g(V)$ , defined in [DLM2], associated with a vertex operator algebra  $V$  and a finite order automorphism  $g$  of  $V$ , to the category of admissible  $g$ -twisted  $V$ -modules.

The associative algebra  $A_n(V)$  plays an important role in the representation theory of vertex operator algebras. One example is the study of logarithmic intertwining operators among generalized modules for a vertex operator algebra. In [HY2], jointly with Y.-Z. Huang, the author proved that the space of logarithmic intertwining operators among generalized modules is naturally isomorphic to the space of homomorphisms between suitable modules for  $A_n(V)$ .

The twisted generalization of the Zhu algebra  $A_g(V)$  was introduced to study twisted modules for a vertex operator algebra with an automorphism of finite order. In [H], Y.-Z. Huang generalized the notion of twisted module for a vertex operator algebra with a finite-order automorphism to the notion of generalized twisted module for a vertex operator algebra with an automorphism of *not necessarily finite order*, using logarithmic conformal field theory. It is natural to define a suitable associative algebra, generalizing the Zhu algebra, associated with a non-finite-order automorphism of  $V$ , and to construct generalized twisted modules in the sense of [H] from certain modules for that associative algebra. For these generalized twisted modules, the twisted vertex operators involve the logarithm of the variable and thus do not have a commutator formula. This motivates us in the present paper to discover a new construction of twisted  $V$ -modules from  $A_g(V)$ -modules without using commutator formula.

The formula for the residues of certain formal series involving iterates of vertex operators, discovered in [HY1], has two undetermined parameters satisfying the lower truncation property. By specializing the parameters to suitable values in the residue formula, we obtain the actions of the relations used to define associative algebras  $A_n(V)$ , as well as  $A_g(V)$ , on the admissible  $V$ -module. This coincidence motivates us to discover the fact that the relations for defining various generalizations of the Zhu algebra are special cases of the residue formula, and hence are implied by the weak associativity, straightforwardly. Based on this fact, we construct functors from the category of modules for the associative algebra to the categories of suitable modules and twisted modules for vertex operator algebras. The most technical part that we prove in this paper is the converse of the fact, that is, that the relations for defining various generalizations of the Zhu algebra imply the residue formula, provided that the lower truncation property and the formula that expresses products of components of vertex operators as linear combinations of iterates of components of vertex operators, mentioned above, hold.

The functors we have constructed in this paper satisfy the same universal property as the functors constructed in [DLM1] and [DLM2]. The importance of our construction is that it allows us to use the method in the present paper to give constructions and prove results in

the cases where there is no commutator formula.

There are various other generalizations of the Zhu algebra associated with a vertex operator algebra in the literature, such as those in [DLM3], [MT] and [VE]. The actions of the relations used to define these associative algebras on certain  $V$ -modules can be obtained from the residue formula by specializing the parameters to suitable values. It is expected that we can use the method in [HY1] and the current paper to give new constructions of functors between module categories for these associative algebras and suitable module categories for the vertex operator algebra.

This paper is organized as follows: We recall the main theorem of [HY1] and deduce some corollaries to motivate this work in Section 2. In Section 3, we recall definitions and properties of the associative algebras  $A_n(V)$  and  $A_g(V)$ . In Section 4 and 5, we use the main theorem to construct a functor from the category of  $A_n(V)$ -modules to the category of admissible  $V$ -modules and prove a universal property of this functor. In Section 6, we apply the main theorem to the twisted module case and construct a functor from the category of  $A_g(V)$ -modules to the category of admissible  $g$ -twisted  $V$ -modules.

**Acknowledgments** I would like to thank Prof. Yi-Zhi Huang for inspiring me to think about this direction and thank Profs. James Lepowsky, Haisheng Li and Katrina Barron for helpful suggestions. I also would like to express my gratitude to Prof. Ping Li for his support during the year 2013–2014.

## 2 An equivalent condition for the associativity

Throughout this paper, we will let  $(V, Y, \mathbf{1}, \omega)$  denote a vertex operator algebra. We state the following theorem from [HY1], using a more general setting:

**Theorem 2.1** *Let  $V$  be a vertex operator algebra,  $W$  a vector space and  $Y_W$  a linear map from  $V \otimes W$  to  $W((x))$ . For  $v \in V$  and  $w \in W$ , as for a  $V$ -module, we denote the image of  $u \otimes w$  under  $Y_W$  by  $Y_W(u, x)w$  and  $\text{Res}_x x^n Y_W(u, x)w$  by  $u_n w$ . Let  $u, v \in V$  and  $w \in W$  and let  $k, l \in \mathbb{R}$  such that*

$$v_n w = 0 \quad \text{for } n \geq k \tag{2.1}$$

and

$$u_n w = 0 \quad \text{for } n \geq l. \tag{2.2}$$

Then the weak associativity for  $W$  in the sense that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_2 + x_0)^l Y_W(Y(u, x_0) v, x_2) w \tag{2.3}$$

is equivalent to the following two properties: For any  $p \in l + \mathbb{Z}$ ,  $q \in k + \mathbb{Z}$ ,

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} (x_0 + x_2)^p x_2^q Y_W(u, x_0 + x_2) Y_W(v, x_2) w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} f(x_0, x_2) x_2^q (x_2 + x_0)^l Y_W(Y(u, x_0) v, x_2) w, \end{aligned} \tag{2.4}$$

where

$$f(x_0, x_2) = \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^i, \quad (2.5)$$

and

$$\text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l-i} x_2^{q+i} (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2)w = 0 \quad (2.6)$$

for  $i \geq k - q$ .

*Proof.* The statement is slightly different from Theorem 3.1 in [HY1] because here  $k, l$  can be real numbers instead of only integers. The proof for “only if” part is the same as [HY1], we give a proof for the “if” part here.

The Laurent polynomial  $p(x_0, x_2)$  is in fact the first  $k - q$  terms of the formal series  $(x_0 + x_2)^{p-l}$ . But from (2.6), we obtain

$$\text{Res}_{x_0} \text{Res}_{x_2} ((x_0 + x_2)^{p-l} - f(x_0, x_2)) x_2^q ((x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2))w = 0. \quad (2.7)$$

Combining (2.4) and (2.7), we obtain

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} (x_0 + x_2)^{p-l} x_2^q ((x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2))w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} (x_0 + x_2)^{p-l} x_2^q ((x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2))w. \end{aligned}$$

On the other hand, we have

$$\text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l-i} x_2^{q+i} (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w = 0 \quad (2.8)$$

for  $i \geq k - q$ . From (2.6) and (2.8), we obtain

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l-i} x_2^{q+i} (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l-i} x_2^{q+i} (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2))w \end{aligned} \quad (2.9)$$

for  $i \geq k - q$ . Combining (2.5) and (2.9), we obtain

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^{q+i} (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^{q+i} (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2))w. \end{aligned} \quad (2.10)$$

We now use induction on  $k - q - 1$  to prove

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l} x_2^q (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l} x_2^q (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2))w \end{aligned} \quad (2.11)$$

for  $p \in \mathbb{Z}$  and  $q < k$ . When  $k - q - 1 = 0$ , (2.10) becomes

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l} x_2^q (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l} x_2^q (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2))w. \end{aligned}$$

Assume that (2.11) holds when  $0 \leq k - q - 1 < n$ . When  $k - q - 1 = n$ ,  $0 \leq k - q - i - 1 < n$  for  $i = 1, \dots, n = k - q - 1$ . Since  $p$  is arbitrary, we can replace  $p$  by  $p - i$  for any  $i \in \mathbb{Z}$  in (2.11). Thus by the induction assumption,

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l-i} x_2^{q+i} (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l-i} x_2^{q+i} (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2) w. \end{aligned} \quad (2.12)$$

for  $i = 1, \dots, n = k - q - 1$ . From (2.12) for  $i = 1, \dots, n = k - q - 1$  and (2.10), we obtain

$$\begin{aligned} & \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l} x_2^q ((x_0 + x_2)^l (Y_W(u, x_0 + x_2) Y_W(v, x_2)) w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^{q+i} (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w \\ & \quad - \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=1}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^{q+i} (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^{q+i} (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2) w \\ & \quad - \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=1}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^{q+i} (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2) w \\ &= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{p-l} x_2^q (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2) w \end{aligned}$$

proving (2.11) in this case.

Taking  $i = 0$  in (2.6), we see that (2.11) also holds for  $p \in \mathbb{Z}$  and  $q \geq k$ . Thus (2.11) holds for  $p, q \in \mathbb{Z}$ . But this means that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2) w,$$

that is, (2.3) holds. ■

The following theorem from [LL] says that the Jacobi identity for the module follows from weak associativity for the module and skew symmetry for the vertex operator algebra:

**Theorem 2.2 ([LL])** *Let  $(V, Y, \mathbf{1})$  be a triple that satisfies all the axioms in the definition of the notion of vertex algebra, in particular the skew symmetry property. Let  $W$  be a vector space and let  $Y_W(\cdot, x)$  be a linear map from  $V$  to  $(\text{End } W)[[x, x^{-1}]]$  such that  $Y_W(\mathbf{1}, x) = 1$  and  $Y_W(v, x)w \in W((x))$  for  $v \in V$  and  $w \in W$ . Assume that weak associativity holds for any  $u, v \in V$  and  $w \in W$ , in the sense that there exists  $l \in \mathbb{N}$  (depending on  $u$  and  $w$ ) such that*

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2) w. \quad (2.13)$$

*Then the Jacobi identity holds for  $u, v \in V$  and  $w \in W$ .*

Therefore, by providing the two properties equivalent to the weak associativity for the module, given in Theorem 2.1, as well as the truncation properties and skew symmetry for the vertex operator algebra, we can obtain the major axiom –Jacobi identity for the vertex operator algebra modules:

**Theorem 2.3** *Let  $V$  be a  $\mathbb{Z}$ -graded vertex algebra,  $W = \coprod_{n \in \mathbb{N}} W_{(n)}$  an  $\mathbb{N}$ -graded vector space and  $Y_W$  a linear map from  $V \otimes W$  to  $W((x))$ . For  $v \in V$  and  $w \in W$ , we denote the image of  $u \otimes w$  under  $Y_W$  by  $Y_W(u, x)w$  and  $\text{Res}_x x^n Y_W(u, x)w$  by  $u_n w$ . Assume that  $u_n$  maps  $W_{(k)}$  to  $W_{(k+m-n-1)}$  for  $u \in V_{(m)}$  and  $n \in \mathbb{Z}$  and  $Y_W(\mathbf{1}, x) = 1_W$ . Also assume that for  $u, v \in V$ ,  $w \in W$ , there exist  $k, l \in \mathbb{Z}$  such that (2.1) and (2.2) hold, and for  $p, q \in \mathbb{Z}$ ,  $u, v \in V$ ,  $w \in W$ , the formulas (2.4) and (2.6) hold. Then  $(W, Y_W)$  is an admissible  $V$ -module. ■*

By a similar proof as Theorem 2.2, we can also show the twisted analogue of Theorem 2.2:

**Theorem 2.4** *Let  $(V, Y, \mathbf{1})$  be  $\mathbb{Z}$ -graded vertex algebra. Suppose that  $V$  has an automorphism  $g$  of order  $T$  and  $V$  has an eigenspace decomposition with respect to the action of  $g$  as*

$$V = \coprod_{r=0}^{T-1} V^r,$$

where

$$V^r = \{v \in V | gv = e^{2\pi ir/T} v\}.$$

Let  $W$  be a vector space and let  $Y_W(\cdot, x)$  be a linear map from  $V$  to  $(\text{End } W)[[x^{\frac{1}{T}}, x^{-\frac{1}{T}}]]$  such that  $Y_W(\mathbf{1}, x) = 1$  and  $Y_W(v, x)w \in W((x^{\frac{1}{T}}))$  for  $v \in V$  and  $w \in W$ . Assume that weak associativity holds for any  $u \in V^r$ ,  $v \in V$  and  $w \in W$ , in the sense that there exists  $l \in \frac{r}{T} + \mathbb{N}$  (depending on  $u$  and  $w$ ) such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_2 + x_0)^l Y_W(Y(u, x_0) v, x_2) w. \quad (2.14)$$

Then the twisted Jacobi identity holds for  $u, v \in V$  and  $w \in W$ .

**Theorem 2.5** *Let  $V$  be a  $\mathbb{Z}$ -graded vertex algebra with a finite order automorphism  $g$ ,  $W = \coprod_{n \in \frac{1}{T}\mathbb{N}} W_{(n)}$  be a  $\frac{1}{T}\mathbb{N}$ -graded vector space and  $Y_W$  be a linear map from  $V \otimes W$  to  $W((x))$ . For  $u \in V^r$  and  $w \in W$ , we denote the image of  $u \otimes w$  under  $Y_W$  by  $Y_W(u, x)w$  and  $\text{Res}_x x^n Y_W(u, x)w$  by  $u_n w$ . Assume that  $u_n$  maps  $W_{(k)}$  to  $W_{(k+m-n-1)}$  for  $u \in V_{(m)}$  and  $n \in \frac{r}{T} + \mathbb{Z}$  and  $Y_W(\mathbf{1}, x) = 1_W$ . Also assume that for  $u, v \in V$ ,  $w \in W$ , there exist  $k, l \in \frac{1}{T}\mathbb{Z}$  such that (2.1) and (2.2) hold, and for  $p \in l + \mathbb{Z}$ ,  $q \in k + \mathbb{Z}$ ,  $u, v \in V$ ,  $w \in W$ , the formulas (2.4) and (2.6) hold. Then  $(W, Y_W)$  is an admissible  $g$ -twisted  $V$ -module. ■*

The component form of (2.6) is

$$\sum_{j=0}^{\infty} \binom{l}{j} (u_{j+p-l-i} v)_{q-j+l+i} w = 0 \quad (2.15)$$

for  $i \geq k - q$ . Set  $N = p + q + 2$  and  $m = p - l - i$ , (2.15) becomes

$$\sum_{j=0}^{\infty} \binom{l}{j} (u_{j+m}v)_{N-j-m-2}w = 0 \quad (2.16)$$

for  $m \leq N - k - l - 2$ . We set  $o(a) = a(\text{wt } a - 1)$  for homogeneous  $a \in V$ . Then by taking  $N = \text{wt } u + \text{wt } v$  in (2.16), we have

$$o\left(\sum_{j=0}^{\infty} \binom{l}{j} u_{j+m}v\right)w = 0, \quad (2.17)$$

where  $m \leq \text{wt } u + \text{wt } v - k - l - 2$ .

**Corollary 2.6** *Let  $W$  be an admissible  $V$ -module,  $u, v \in V$  and  $w \in W_{(n)}$  for  $n \in \mathbb{N}$ . Then*

$$o\left(\sum_{j=0}^{\text{wt } u+n} \binom{\text{wt } u+n}{j} u_{j+m}v\right)w = 0,$$

where  $m \leq -2n - 2$ .

*Proof.* In equation (2.17), let  $k = \text{wt } v + n$  and  $l = \text{wt } u + n$ . ■

**Corollary 2.7** *Suppose that  $V$  has an automorphism  $g$  of order  $T$  and an eigenspace decomposition with respect to the action of  $g$  as*

$$V = \coprod_{r=0}^{T-1} V^r,$$

where

$$V^r = \{v \in V \mid gv = e^{2\pi ir/T}v\}.$$

Let  $W$  be an admissible  $g$ -twisted  $V$ -module,  $u \in V^r, v \in V$  and  $w \in W_{(0)}$ . Then

$$o\left(\sum_{j=0}^{\infty} \binom{\text{wt } u - 1 + \delta_r + \frac{r}{T}}{j} u_{j+m}v\right)w = 0$$

for  $m \leq -\delta_r - 1$ .

*Proof.* In equation (2.17), let  $k = \text{wt } v - \frac{r}{T}$  and  $l = \text{wt } u - 1 + \delta_r + \frac{r}{T}$ . ■

The product formula (2.4) will be used to calculate the product of operator components acting on the module:

**Lemma 2.8** *In the setting of Theorem 2.1. The product formula (2.4) is equivalent to*

$$u_p v_q w = \text{Res}_{x_0} \text{Res}_{x_2} \left( \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^i \right) x_2^q (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2)w. \quad (2.18)$$

### 3 Zhu algebra

In this section, we will recall the definition and some properties of the Zhu algebra introduced in [Z], a generalization of the Zhu algebra  $A_n(V)$  defined in [DLM1] for  $n \in \mathbb{N}$  and twisted generalization of the Zhu algebra  $A_g(V)$  defined in [DLM2] for a finite order automorphism  $g$  of  $V$ .

We first recall the definition of the Zhu algebra  $A(V)$  for a vertex operator algebra  $V$ .

**Definition 3.1 ([Z])** Let  $O(V)$  be the subspace of  $V$  spanned by elements of the form

$$\text{Res}_x \frac{(1+x)^{\text{wt}} {}^u Y(u, x)v}{x^2}$$

for homogeneous  $u, v \in V$ . Zhu algebra  $A(V)$  is defined to be the quotient space  $V/O(V)$ .

The Zhu algebra was then generalized to an associative algebra  $A_n(V)$  for  $n \in \mathbb{N}$  and to a twisted Zhu algebra  $A_g(V)$  for an automorphism  $g$  of  $V$ .

**Definition 3.2 ([DLM1])** For  $n \in \mathbb{N}$ , let  $O_n(V)$  be the subspace of  $V$  spanned by elements of the form

$$\text{Res}_x \frac{(1+x)^{\text{wt}} {}^{u+n} Y(u, x)v}{x^{2n+2}}$$

for homogeneous  $u, v \in V$ .  $A_n(V)$  is defined to be the quotient space  $V/O_n(V)$ .

The subspace  $O_n(V)$  is a two-sided ideal of  $V$  and  $A_n(V)$  is an associative algebra under the multiplication  $*$  defined by

$$u *_n v = \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_x \frac{(1+x)^{\text{wt}} {}^{u+n} Y(u, x)v}{x^{n+m+1}},$$

for homogeneous  $u, v \in V$ , and for general  $u, v \in V$ ,  $*$  is defined by linearity. Also, for every homogeneous element  $u \in V$  and  $m \geq k \geq 0$ , elements of the form

$$\text{Res}_x \frac{(1+x)^{\text{wt}} {}^{u+n+k} Y(u, x)v}{x^{m+2n+2}} \quad (3.1)$$

lie in  $O_n(V)$ .

Let  $W$  be an admissible  $V$ -module and let  $\Omega_m(W)$  denote the subspace consisting of  $m$ th lowest weight vectors in  $W$ , that is

$$\Omega_m(W) = \{w \in W \mid u_n w = 0 \text{ if } \text{wt } u_n < -m\}.$$

It was shown in [DLM1] that  $\Omega_n(W)$  is an  $A_n(V)$ -module via the action  $o(v + O_n(V)) = v_{\text{wt } v-1}$  for  $v \in V$ . The first necessary condition to prove is that  $o(O_n(V))$  annihilates  $\Omega_n(W)$ . Corollary 2.6 gives an alternating proof of this condition and also motivates us to give another construction of functors between categories of  $A_n(V)$ -modules and categories of admissible  $V$ -modules (see Section 4 and 5 for the detail).



**Definition 3.3 ([DLM2])** For an automorphism  $g$  of  $V$  of finite order  $T$ , let  $O_g(V)$  be the subspace of  $V$  spanned by elements of the form

$$\text{Res}_x \frac{(1+x)^{\text{wt } u-1+\delta_r+\frac{r}{T}} Y(u, x)v}{x^{1+\delta_r}}$$

for homogeneous  $u \in V^r, v \in V$ , where

$$V^r = \{v \in V \mid gv = e^{2\pi ir/T} v\}.$$

The twisted Zhu algebra  $A_g(V)$  is defined to be the quotient space  $V/O_g(V)$ .

The subspace  $O_g(V)$  is a two-sided ideal of  $V$  and  $A_g(V)$  is an associative algebra under the multiplication  $*$  defined by

$$u *_g v = \begin{cases} \text{Res}_x \frac{(1+x)^{\text{wt } u} Y(u, x)v}{x} & \text{if } r = 0, \\ 0 & \text{if } r > 0. \end{cases}$$

for homogeneous  $u \in V^r, v \in V$ , and for general  $u, v \in V$ ,  $*_g$  is defined by linearity. Also, for every homogeneous element  $u \in V^r$  and  $m \geq k \geq 0$ , elements of the form

$$\text{Res}_x \frac{(1+x)^{\text{wt } u-1+\delta_r+\frac{r}{T}+k} Y(u, x)v}{x^{m+\delta_r+1}} \quad (3.2)$$

lie in  $O_g(V)$ .

The subspace  $V^r \subset O_g(V)$  for  $r \neq 0$  and  $A_g(V)$  is a quotient of  $A(V^0)$ . Therefore, an  $A_g(V)$ -module can be lift to an  $A(V^0)$ -module.

Let  $W$  be an admissible twisted  $V$ -module and let  $\Omega(W)$  denote the subspace consisting of lowest weight vectors in  $W$ , that is

$$\Omega(W) = \{w \in W \mid u_n w = 0 \text{ if } \text{wt } u_n < 0\}.$$

It was shown in [DLM2] that  $\Omega(W)$  is an  $A_g(V)$ -module via the action  $o(v+O_g(V)) = v_{\text{wt } v-1}$  for  $v \in V^0$ . The first thing to show is that  $o(O_g(V))$  annihilates  $\Omega(W)$ . Corollary 2.7 gives an alternating proof of this fact and also motivates us to give another construction of functors between categories of  $A_g(V)$ -modules and categories of admissible  $g$ -twisted  $V$ -modules (see Section 6 for the detail).

## 4 A functor $S_n$ from the category of $A_n(V)$ -modules to the category of $V$ -modules

In the remaining part of this paper, we will assume the vertex operator algebra  $V$  is  $\mathbb{N}$ -graded.

In this section, we will start from an  $A_n(V)$ -module  $W$  and construct a vector space  $S_n(W)$  with a linear map  $Y_{S_n(W)}$  from  $V \otimes S_n(W)$  to  $S_n(W)((x))$ . Then we use Theorem 2.1

to show that weak associativity holds for the pair  $(S_n(W), Y_{S_n(W)})$  and hence the pair is an admissible  $V$ -module by Theorem 2.3.

Consider the affinization  $V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}]$  of  $V$  and the tensor algebra  $T(V[t, t^{-1}])$  generated by  $V[t, t^{-1}]$ . For simplicity, we shall denote  $u \otimes t^m$  for  $u \in V$  and  $m \in \mathbb{Z}$  by  $u(m)$  and we shall omit the tensor product sign  $\otimes$  when we write an element of  $T(V[t, t^{-1}])$ . Thus  $T(V[t, t^{-1}])$  is spanned by elements of the form  $u_1(m_1) \cdots u_k(m_k)$  for  $u_i \in V$  and  $m_i \in \mathbb{Z}$ ,  $i = 1, \dots, k$ .

Consider  $T(V[t, t^{-1}]) \otimes W$ . Again for simplicity we shall omit the tensor product sign. So  $T(V[t, t^{-1}]) \otimes W$  is spanned by elements of the form  $u_1(m_1) \cdots u_k(m_k)w$  for  $u_i \in V$ ,  $m_i \in \mathbb{Z}$ ,  $i = 1, \dots, k$  and  $w \in W$  and for any  $u \in V$ ,  $m \in \mathbb{Z}$ ,  $u(m)$  acts from the left on  $T(V[t, t^{-1}]) \otimes W$ . For homogeneous  $u_i \in V$ ,  $m_i \in \mathbb{Z}$ ,  $i = 1, \dots, k$  and  $w \in W$ , we define the degree of elements in  $T(V[t, t^{-1}]) \otimes W$  as follows:

$$\deg u_1(m_1) \cdots u_k(m_k)w = (\text{wt } u_1 - m_1 - 1) + \cdots + (\text{wt } u_k - m_k - 1) + n.$$

For any  $u \in V$ , let

$$Y_t(u, x) : T(V[t, t^{-1}]) \otimes W \longrightarrow T(V[t, t^{-1}]) \otimes W[[x, x^{-1}]]$$

be defined by

$$Y_t(u, x) = \sum_{m \in \mathbb{Z}} u(m)x^{-m-1}.$$

For a homogeneous element  $u \in V$ , let

$$o_t(u) = u(\text{wt } u - 1).$$

Using linearity, we extend  $o_t(u)$  to non-homogeneous  $u$ .

Let  $\rho : A_n(V) \rightarrow \text{End } W$  be a representation of associative algebra  $A_n(V)$ . Let  $\mathcal{I}$  be the  $\mathbb{Z}$ -graded  $T(V[t, t^{-1}])$ -submodule of  $T(V[t, t^{-1}]) \otimes W$  generated by elements of the forms  $u(m)w$  ( $u \in V$ ,  $\text{wt } u - m - 1 + \deg w < 0$ ,  $w \in T(V[t, t^{-1}]) \otimes W$ ),  $o_t(u)w - \rho(u + O_n(V))w$  ( $u \in V$ ,  $w \in W$ ) and

$$\begin{aligned} & u(p)v(q)w - \sum_{i=0}^{\text{wt } v + \deg w + n - q - 1} \sum_{j=0}^{\text{wt } u + \deg w + n} \binom{p - \text{wt } u - \deg w - n}{i} \\ & \cdot \binom{\text{wt } u + \deg w + n}{j} (u_{p - \text{wt } u - \deg w - i + j - n} v)(q + \text{wt } u + \deg w + i - j + n)w \end{aligned} \quad (4.1)$$

( $u, v \in V$ ,  $q \in \mathbb{Z}$  such that  $\text{wt } v - q - 1 + \deg w \geq 0$ ,  $w \in T(V[t, t^{-1}]) \otimes W$ ). Note that relation (4.1) is derived from formula (2.18) (also (2.4)) when  $k = \text{wt } v + \deg w + n$  and  $l = \text{wt } u + \deg w + n$ .

Let  $\tilde{S}_n(W) = T(V[t, t^{-1}]) \otimes W / \mathcal{I}$ . Then  $\tilde{S}_n(W)$  is also a  $\mathbb{Z}$ -graded  $T(V[t, t^{-1}])$ -module. In fact, by definition of  $\mathcal{I}$ , we see that  $\tilde{S}_n(W)$  is spanned by elements of the form  $u(m)w + \mathcal{I}$

for homogeneous  $u \in V$ ,  $m \in \mathbb{Z}$  such that  $m < \text{wt } u + n - 1$  and  $w \in W$ . In particular, we see that  $\tilde{S}_n(W)$  has an  $\mathbb{N}$ -grading. Note that  $\mathcal{I} \cap W = \{0\}$ ,  $W$  can be embedded into  $\tilde{S}_n(W)$  and  $(\tilde{S}_n(W))_n = W$ .

Let  $\mathcal{J}$  be the  $\mathbb{N}$ -graded  $T(V[t, t^{-1}])$ -submodule of  $\tilde{S}_n(W)$  generated by

$$\sum_{j=0}^{\text{wt } u + \deg w + n} \binom{\text{wt } u + \deg w + n}{j} (u_{j+m}v)(N - j - m - 2)w \quad (4.2)$$

( $u, v \in V$ ,  $w \in \tilde{S}_n(W)$ ,  $N \in \mathbb{Z}$ ,  $m \leq N - 2 - \text{wt } u - \text{wt } v - 2n - 2 \deg w$ ). Note that relation (4.2) comes from the formula (2.16) by specializing  $k = \text{wt } v + \deg w + n$  and  $l = \text{wt } u + \deg w + n$ .

Let  $S_n(W) = \tilde{S}_n(W)/\mathcal{J}$ . Then  $S_n(W)$  is also an  $\mathbb{N}$ -graded  $T(V[t, t^{-1}])$ -module. We can still use elements of  $T(V[t, t^{-1}]) \otimes W$  to represent elements of  $S_n(W)$ . But note that these elements now satisfy relations. We equip  $S_n(W)$  with the vertex operator map

$$Y_{S_n(W)} : V \otimes S_n(W) \longrightarrow S_n(W)[[x, x^{-1}]]$$

given by

$$u \otimes w \rightarrow Y(u, x)w = Y_t(u, x)w.$$

**Theorem 4.1** *The pair  $(S_n(W), Y_{S_n(W)})$  is an admissible  $V$ -module.*

*Proof.* As in  $\tilde{S}_n(W)$ , for  $u \in V$  and  $w \in S_n(W)$ , we also have  $u(m)w = 0$  when  $m > \text{wt } u + \deg w - 1$ . Clearly,

$$Y(\mathbf{1}, x) = I_{S_n(W)},$$

where  $I_{S_n(W)}$  is the identity operator on  $S_n(W)$ .

Since  $S_n(W)$  satisfies the product formula and iterate formula in Theorem 2.1, where we specialize  $k = \text{wt } v + \deg w + n$  and  $l = \text{wt } u + \deg w + n$ , by definition of  $\mathcal{I}$  and  $\mathcal{J}$ ,  $S_n(W)$  is an admissible weak  $V$ -module.  $\blacksquare$

Let  $W_1$  and  $W_2$  be  $A_n(V)$ -modules and  $f : W_1 \rightarrow W_2$  a module map. Then  $f$  induces a linear map from  $T(V[t, t^{-1}]) \otimes W_1$  to  $T(V[t, t^{-1}]) \otimes W_2$ . By definition, this induced linear map in turn induces a linear map  $S_n(f)$  from  $S_n(W_1)$  to  $S_n(W_2)$ . Since  $Y_{S_n(W_1)}$  and  $Y_{S_n(W_2)}$  are induced by  $Y_t$  on  $T(V[t, t^{-1}]) \otimes W_1$  and  $T(V[t, t^{-1}]) \otimes W_2$ , respectively, we have

$$S_n(f)(Y_{S_n(W_1)}(u, x)w_1) = Y_{S_n(W_2)}(u, x)S_n(f)(w_1)$$

for  $u \in V$  and  $w_1 \in S_n(W_1)$ . Thus  $S_n(f)$  is a module map. The following result is now clear:

**Corollary 4.2** *Let  $V$  be an  $\mathbb{N}$ -graded vertex algebra. Then the correspondence sending an  $A_n(V)$ -module  $W$  to an admissible  $V$ -module  $(S_n(W), Y_{S_n(W)})$  and an  $A_n(V)$ -module map  $W_1 \rightarrow W_2$  to a  $V$ -module map  $S_n(f) : S_n(W_1) \rightarrow S_n(W_2)$  is a functor from the category of  $A_n(V)$ -modules to the category of admissible  $V$ -modules.*

## 5 A universal property for $S_n$

In this section, we prove that  $S_n$  satisfies a natural universal property and thus is the same as the functor constructed in [DLM1]. In particular, we achieve our goal of constructing admissible  $V$ -modules from  $A_n(V)$ -modules without dividing relations corresponding to the commutator formula for weak modules.

We use the following two lemmas to prove that  $\mathcal{J} \cap W = 0$  in  $\tilde{S}_n(W)$  and hence  $(S_n(W))_n = W$ .

**Lemma 5.1** *Let  $M \in \mathbb{N}$  and  $M \leq n$ . Then in  $\tilde{S}_n(W)$ ,*

$$a(\text{wt } a - M - 1) \left( \sum_{j=0}^{\text{wt } u+n} \binom{\text{wt } u+n}{j} (u_{j+m}v)(\text{wt } u_{j+m}v + M - 1) \right) w = 0,$$

where  $w \in W$  and  $m \leq M - 3n - 2$ .

*Proof.* We apply formula (2.18) to

$$a(\text{wt } a - M - 1) \left( \sum_{j=0}^{\text{wt } u+n} \binom{\text{wt } u+n}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) w$$

by specifying

$$\begin{aligned} p &= \text{wt } a - M - 1 \\ q &= \text{wt } u + \text{wt } v - j - m + M - 2 \\ k &= \text{wt } u + \text{wt } v - j - m - 1 + 2n \\ l &= \text{wt } a + 2n, \end{aligned}$$

we have

$$\begin{aligned}
& a(\text{wt } a - M - 1) \left( \sum_{j=0}^{\text{wt } u+n} \binom{\text{wt } u+n}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) w \\
= & \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{2n-M} \sum_{j=0}^{\text{wt } u+n} \binom{-M-2n-1}{i} \binom{\text{wt } u+n}{j} x_0^{-M-2n-i-1} x_2^{\text{wt } u + \text{wt } v - j - m + M - 2 + i} \\
& \cdot (x_2 + x_0)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(Y(a, x_0)(u_{j+m}v), x_2)w \\
= & \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{2n-M} \binom{-M-2n-1}{i} (1+y)^{\text{wt } u+n} y^m x_0^{-M-2n-i-1} \\
& \cdot (x_2 + x_0)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(Y(a, x_0)x_2^{L(0)+M-1+i}Y(u, y)v, x_2)w \\
= & \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{2n-M} \binom{-M-2n-1}{i} (1+y)^{\text{wt } u+n} y^m x_0^{-M-2n-i-1} x_2^{\text{wt } u + \text{wt } v + M - 1 + i} \\
& \cdot (x_2 + x_0)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(Y(a, x_0)Y(u, x_2y)v, x_2)w \\
= & \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{2n-M} \binom{-M-2n-1}{i} (1+y)^{\text{wt } u+n} y^m x_0^{-M-2n-i-1} x_2^{\text{wt } u + \text{wt } v + M - 1 + i} \\
& \cdot (x_2 + x_0)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(Y(u, x_2y)Y(a, x_0)v, x_2)w \\
& + \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{2n-M} \binom{-M-2n-1}{i} (1+y)^{\text{wt } u+n} y^m x_0^{-M-2n-i-1} \\
& \cdot x_2^{\text{wt } u + \text{wt } v + M - 1 + i} (x_2 + x_0)^{\text{wt } a+2n} x_0^{-1} \delta\left(\frac{x_2y + x_1}{x_0}\right) Y_{\tilde{S}_n(W)}(Y(Y(a, x_1)u, x_2y)v, x_2)w.
\end{aligned}$$

By examining the monomials in  $y$  in the first term of the right-hand side, we know that the first term of the right-hand side is a sum of elements of form

$$o \left( \sum_{j=0}^{\text{wt } u+n} \binom{\text{wt } u+n}{j} (u_{j+m}\tilde{a}) \right) w$$

with  $m \leq M - 3n - 2$  for some  $\tilde{a} \in V$ . Hence the first term is an action of elements in  $O_n(V)$  on  $w$  which is 0. We only need to prove the second term is also an action of a sum

of elements of form  $O_n(V)$  on  $w$ . The second term equals

$$\begin{aligned}
& \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{2n-M} \binom{-M-2n-1}{i} (1+y)^{\text{wt } u+n} y^m x_0^{-M-2n-i-1} \\
& \quad \cdot x_2^{\text{wt } u+\text{wt } v+M-1+i} (x_2+x_0)^{\text{wt } a+2n} x_0^{-1} \delta\left(\frac{x_2 y + x_1}{x_0}\right) Y_{\tilde{S}_n(W)}(Y(Y(a, x_1)u, x_2 y)v, x_2)w \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{2n-M} \binom{-M-2n-1}{i} (1+y)^{\text{wt } u+n} y^m (x_2 y + x_1)^{-M-2n-i-1} \\
& \quad \cdot x_2^{\text{wt } u+\text{wt } v+M-1+i} (x_2+x_2 y+x_1)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(Y(Y(a, x_1)u, x_2 y)v, x_2)w \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^{2n-M} \binom{-M-2n-1}{i} (1+y)^{\text{wt } u+\text{wt } a+3n+1} y^m (x_2 y + x_3(1+y))^{-M-2n-i-1} \\
& \quad \cdot x_2^{\text{wt } u+\text{wt } v+M-1+i} (x_2+x_3)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(Y(Y(a, (1+y)x_3)u, x_2 y)v, x_2)w \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^{2n-M} \sum_{j=0}^{\infty} \binom{-M-2n-1}{i} \binom{-M-2n-i-1}{j} (1+y)^{\text{wt } u+\text{wt } a+3n+1+j} \\
& \quad \cdot y^{m-M-2n-i-j-1} x_2^{\text{wt } u+\text{wt } v-2n-j-2} x_3^j (x_2+x_3)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(Y(Y(a, (1+y)x_3)u, x_2 y)v, x_2)w \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^{2n-M} \sum_{j=0}^{\infty} \binom{-M-2n-1}{i} \binom{-M-2n-i-1}{j} x_2^{\text{wt } u+\text{wt } v-2n-j-2} x_3^j \\
& \quad \cdot (x_2+x_3)^{\text{wt } a+2n} Y_{\tilde{S}_n(W)}(y^{m-M-2n-i-j-1} Y((1+y)^{L(0)+3n+1+j} Y(a, x_3)u, x_2 y)v, x_2)w.
\end{aligned}$$

Since  $m \leq M - 3n - 2$  and  $i, j \geq 0$ , we obtain that it is an action of elements of the form

$$\text{Res}_y y^{m'} Y(1+y)^{L(0)+3n+1+j} (Y(a, x_3)u, y)v$$

on  $w$ , where  $m' = m - M - 2n - i - j - 1 \leq -5n - i - j - 3$ . Apparently, this is an element in  $O_n(V)$ , the action equals 0.  $\blacksquare$

We proceed to prove the next proposition:

**Proposition 5.2** *Let  $M, N \in \mathbb{Z}$  such that  $0 \leq N + n$  and  $M \leq N + n$ . Then in  $\tilde{S}_n(W)$ ,*

$$\begin{aligned}
& a(\text{wt } a - M + N - 1) \\
& \cdot \left( \sum_{j=0}^{\text{wt } u+N+2n} \binom{\text{wt } u + N + 2n}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) \\
& \cdot b(\text{wt } b - N - 1)w = 0
\end{aligned} \tag{5.1}$$

where  $m \leq M - 2N - 4n - 2$  and  $w \in W$ .

*Proof.* By Lemma 5.1, it suffices to show that the element

$$\left( \sum_{j=0}^{\text{wt } u + N + 2n} \binom{\text{wt } u + N + 2n}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) \cdot b(\text{wt } b - N - 1)w$$

for  $m \leq M - 2N - 4n - 2$ , is of the form

$$\sum_{j=0}^{\text{wt } \tilde{u} + n} \binom{\text{wt } \tilde{u} + n}{j} (\tilde{u}_{j+m}\tilde{v})(\text{wt } \tilde{u} + \text{wt } \tilde{v} - j - m + M - N - 2)w$$

for  $m \leq M - N - 3n - 2$ ,  $\tilde{u}, \tilde{v} \in V$ .

We apply formula (2.18) to

$$\left( \sum_{j=0}^{\text{wt } u + N + 2n} \binom{\text{wt } u + N + 2n}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) b(\text{wt } b - N - 1)w$$

by specifying

$$\begin{aligned} p &= \text{wt } u + \text{wt } v - j - m + M - 2 \\ q &= \text{wt } b - N - 1 \\ k &= \text{wt } b + 2n \\ l &= \text{wt } u + \text{wt } v - j - m - 1 + 2n, \end{aligned}$$

we have

$$\begin{aligned}
& \sum_{j=0}^{\text{wt } u+N+2n} \binom{\text{wt } u + N + 2n}{j} (u_{j+m}v) (\text{wt } u_{j+m}v + M - 1) b (\text{wt } b - N - 1) w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_y \sum_{i=0}^{2n+N} \binom{M-2n-1}{i} x_0^{M-2n-i-1} x_2^{\text{wt } b-N-1+i} (1+y)^{\text{wt } u+N+2n} y^m \\
&\quad \cdot Y_{\tilde{S}_n(W)}(Y((x_2+x_0)^{L(0)+2n} Y(u, y)v, x_0)b, x_2)w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_y \sum_{i=0}^{2n+N} \binom{M-2n-1}{i} x_0^{M-2n-i-1} x_2^{\text{wt } b-N-1+i} (1+y)^{\text{wt } u+N+2n} y^m \\
&\quad \cdot (x_2+x_0)^{\text{wt } u+\text{wt } v+2n} Y_{\tilde{S}_n(W)}(Y(Y(u, (x_2+x_0)y)v, x_0)b, x_2)w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^{2n+N} \binom{M-2n-1}{i} x_0^{M-2n-i-1} x_2^{\text{wt } b-N-1+i} (x_2+x_0+x_3)^{\text{wt } u+N+2n} \\
&\quad \cdot x_3^m (x_2+x_0)^{\text{wt } v-N-m-1} Y_{\tilde{S}_n(W)}(Y(Y(u, x_3)v, x_0)b, x_2)w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{2n+N} \binom{M-2n-1}{i} x_0^{M-2n-i-1} x_2^{\text{wt } b-N-1+i} (x_2+x_1)^{\text{wt } u+N+2n} \\
&\quad \cdot (x_1-x_0)^m (x_2+x_0)^{\text{wt } v-N-m-1} Y_{\tilde{S}_n(W)}(Y(u, x_1)Y(v, x_0)b, x_2)w \\
&\quad - \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{2n+N} \binom{M-2n-1}{i} x_0^{M-2n-i-1} x_2^{\text{wt } b-N-1+i} (x_2+x_1)^{\text{wt } u+N+2n} \\
&\quad \cdot (-x_0+x_1)^m (x_2+x_0)^{\text{wt } v-N-m-1} Y_{\tilde{S}_n(W)}(Y(v, x_0)Y(u, x_1)b, x_2)w
\end{aligned}$$

By examining the monomials in  $x_0$  in the second term of the right-hand side, it is a sum of elements of the form

$$\sum_{j=0}^{\text{wt } v-N-m-1} \binom{\text{wt } v - N - m - 1}{j} (v_{j+m'}\tilde{u}) (\text{wt } v + \text{wt } \tilde{u} - j - m + M - N - 2)$$

acting on  $w$  for some  $\tilde{u} \in V$  and  $m' \leq m + M - 2n - 1$ . Since

$$-N - m - 1 \geq -M + N + 4n + 1 \geq n,$$

this element is of the form

$$\sum_{j=0}^{\text{wt } v+n} \binom{\text{wt } v + n}{j} (v_{j+m''}\tilde{u}) (\text{wt } v + \text{wt } \tilde{u} - j - m + M - N - 2)$$

for  $m'' = m' - N - m - n - 1 \leq M - N - 3n - 2$  and  $\tilde{u} \in V$ . It is easy to see by examining the monomials in  $x_1$  in first term of the right-hand side that the first term is also a sum of elements of this form. ■



The following theorem is an easy consequence of Proposition 5.2:

**Theorem 5.3** *In  $\tilde{S}_n(W)$ ,*

$$\mathcal{J} \cap W = 0.$$

*The embedding of  $W$  to  $\tilde{S}_n(W)$  induces an injection  $e_W$  of  $A_n(V)$ -modules from  $W$  to  $\Omega_n(S_n(W))$ .*

**Theorem 5.4** *The functor  $S_n$  has the following universal property: Let  $W$  be an  $A_n(V)$ -module. For any admissible  $V$ -module  $\tilde{W}$  and any  $A_n(V)$ -module map  $f : W \rightarrow \Omega_n(\tilde{W})$ , there exists a unique  $V$ -module map  $\tilde{f} : S_n(W) \rightarrow \tilde{W}$  such that  $\tilde{f}|_{e_W(W)} = f \circ e_W^{-1}$ .*

In [DLM1], a functor, denoted by  $\bar{M}_n$ , was constructed explicitly and was proved to satisfy the same universal property above. The following result achieves our goal of constructing this functor without dividing relations corresponding to the commutator formula for weak modules:

**Corollary 5.5** *The functor  $S_n$  is equal to the functor  $\bar{M}_n$  constructed in [DLM1].*

*Proof.* This result follows immediately from the universal property. ■

## 6 A functor $S_g$ from the category of $A_g(V)$ -modules to the category of twisted $V$ -module

In this section, we assume the vertex operator algebra  $V$  has an automorphism  $g$  with finite order  $T$  and an eigenspace decomposition with respect to the action of  $g$  as

$$V = \coprod_{r=0}^{T-1} V^r,$$

where

$$V^r = \{v \in V | gv = e^{2\pi ir/T} v\}.$$

We shall start with an  $A_g(V)$ -module  $W$  and construct a vector space  $S_g(W)$  with a linear map  $Y_{S_g(W)}$  from  $V \otimes S_g(W)$  to  $S_g(W)((x^{\frac{1}{T}}))$ . Then we show that the pair  $(S_g(W), Y_{S_g(W)})$  is an admissible twisted  $V$ -module by verifying that weak associativity holds on the pair.

Consider the affinization  $V^g[t, t^{-1}] = \coprod_{r=0}^{T-1} V^r \otimes t^{r/T} \mathbb{C}[t, t^{-1}]$  of  $V$  and the tensor algebra  $T(V^g[t, t^{-1}])$  generated by  $V^g[t, t^{-1}]$ . For simplicity, we shall denote  $u \otimes t^m$  for  $u \in V^r$  and  $m \in \frac{r}{T} + \mathbb{Z}$  by  $u(m)$  and we shall omit the tensor product sign  $\otimes$  when we write an element of  $T(V^g[t, t^{-1}])$ . Thus  $T(V^g[t, t^{-1}])$  is spanned by elements of the form  $u_1(m_1) \cdots u_k(m_k)$  for  $u_i \in V^{r_i}$  and  $m_i \in \frac{r_i}{T} + \mathbb{Z}$ ,  $r_i = 0, 1, \dots, T-1$  for  $i = 1, \dots, k$ .

Consider  $T(V^g[t, t^{-1}]) \otimes W$ . Again for simplicity we omit the tensor product sign. So  $T(V^g[t, t^{-1}]) \otimes W$  is spanned by elements of the form  $u_1(m_1) \cdots u_k(m_k)w$  for  $u_i \in V^{r_i}$ ,

$m_i \in \frac{r_i}{T} + \mathbb{Z}$ ,  $i = 1, \dots, k$  and  $w \in W$  and for any  $u \in V^r$ ,  $m \in \frac{r}{T} + \mathbb{Z}$ ,  $u(m)$  acts from the left on  $T(V^g[t, t^{-1}]) \otimes W$ . For homogeneous  $u_i \in V^{r_i}$ ,  $m_i \in \frac{r_i}{T} + \mathbb{Z}$ ,  $i = 1, \dots, k$  and  $w \in W$ , we define the degree of elements in  $T(V^g[t, t^{-1}]) \otimes W$  as

$$\deg u_1(m_1) \cdots u_k(m_k)w = (\text{wt } u_1 - m_1 - 1) + \cdots (\text{wt } u_k - m_k - 1).$$

For any  $u \in V^r$ , define

$$Y_t(u, x) : T(V^g[t, t^{-1}]) \otimes W \longrightarrow T(V^g[t, t^{-1}]) \otimes W[[x^{\frac{1}{T}}, x^{-\frac{1}{T}}]]$$

to be

$$Y_t(u, x) = \sum_{m \in \frac{r}{T} + \mathbb{Z}} u(m)x^{-m-1}.$$

For a homogeneous element  $u \in V^0$ , let

$$o_t(u) = u(\text{wt } u - 1).$$

Using linearity, we extend  $o_t(u)$  to non-homogeneous  $u \in V^0$ .

Let  $\rho : A_g(V) \rightarrow \text{End } W$  be a representation of associative algebra  $A_g(V)$ . Let  $\mathcal{I}$  be the  $\mathbb{Z}$ -graded  $T(V^g[t, t^{-1}])$ -submodule of  $T(V^g[t, t^{-1}]) \otimes W$  generated by elements of the forms  $u(m)w$  ( $u \in V$ ,  $\text{wt } u - m - 1 + \deg w < 0$ ,  $w \in T(V^g[t, t^{-1}]) \otimes W$ ),  $o_t(u)w - \rho(u + O_g(V) \cap V^0)w$  ( $u \in V^0$ ,  $w \in W$ ) and

$$\begin{aligned} & u(p)v(q)w - \sum_{i=0}^{\text{wt } v + \deg w - q - 1} \sum_{j=0}^{\infty} \binom{p - \text{wt } u - \deg w - \delta_r - \frac{r}{T}}{i} \binom{\text{wt } u + \deg w + \delta_r + \frac{r}{T}}{j} \\ & \cdot (u_{p - \text{wt } u - \deg w - \delta_r - \frac{r}{T} - i + j}v)(q + \text{wt } u + \deg w + \delta_r + \frac{r}{T} + i - j)w \end{aligned}$$

( $u \in V^r$ ,  $v \in V$ ,  $q \in \mathbb{Z}$  such that  $\text{wt } v - q - 1 + \deg w \geq 0$ ,  $w \in T(V^g[t, t^{-1}]) \otimes W$ ).

Let  $\tilde{S}(W) = T(V^g[t, t^{-1}]) \otimes W / \mathcal{I}$ . Then  $\tilde{S}(W)$  is also a  $\frac{1}{T}\mathbb{Z}$ -graded  $T(V^g[t, t^{-1}])$ -module. In fact, by definition of  $\mathcal{I}$ , we see that  $\tilde{S}(W)$  is spanned by elements of the form  $u(m)w + \mathcal{I}$  for homogeneous  $u \in V^r$ ,  $m \in \frac{r}{T} + \mathbb{Z}$  such that  $m < \text{wt } u - 1$  and  $w \in W$ . In particular, we see that  $\tilde{S}(W)$  has an  $\frac{1}{T}\mathbb{N}$ -grading. Note that  $\mathcal{I} \cap W = \{0\}$ ,  $W$  can be embedded into  $\tilde{S}(W)$  and  $(\tilde{S}(W))_0 = W$ .

Let  $\mathcal{J}$  be the  $\frac{1}{T}\mathbb{N}$ -graded  $T(V^g[t, t^{-1}])$ -submodule of  $\tilde{S}(W)$  generated by

$$\sum_{j=0}^{\infty} \binom{\text{wt } u + \deg w + \delta_r + \frac{r}{T}}{j} (u_{j+m}v)(N - j - m - 2)w$$

( $u \in V^r$ ,  $v \in V^s$ ,  $w \in \tilde{S}(W)$ ,  $N \in \frac{r+s}{T} + \mathbb{Z}$ ,  $m \leq N - 2 - \text{wt } u - \text{wt } v - 2 \deg w - \delta_r - \frac{r}{T}$ ).

**Theorem 6.1** *Let  $\mathcal{J}$  be the  $\frac{1}{T}\mathbb{N}$ -graded  $T(V^g[t, t^{-1}])$ -submodule of  $\tilde{S}(W)$  defined above. Then in  $\tilde{S}(W)$ ,*

$$\mathcal{J} \cap W = 0.$$

*Proof.* It suffices to prove the element of the form

$$\begin{aligned}
& a(\text{wt } a - M + N - 1) \\
& \cdot \left( \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) \\
& \cdot b(\text{wt } b - N - 1)w
\end{aligned} \tag{6.1}$$

equals 0 in  $\tilde{S}(W)$ , where  $a \in V^d$ ,  $u \in V^r$ ,  $v \in V^s$ ,  $b \in V^t$  ( $d + r + s + t \equiv 0 \pmod{T}$ ),  $w \in W$ ,  $M \in \frac{r+s}{T} + \mathbb{Z}$ ,  $N \in \frac{T-t}{T} + \mathbb{Z}$ ,

$$m \leq -N - 2 - (N - M) - \delta_r - \frac{r}{T}.$$

We will prove the claim in the following cases:

(i)  $N < 0$ . Since the element  $b(\text{wt } b - N - 1)w$  has negative grading  $N$ , the expression (6.1) lies in  $\mathcal{I}$  and is 0 in  $\tilde{S}(W)$ ;

(ii)  $N \geq 0$  and  $M > N$ . Applying formula (2.18) to the expression

$$\left( \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) b(\text{wt } b - N - 1)w,$$

it becomes an element with negative grading  $N - M$ , which lies in  $\mathcal{I}$  and hence equals 0 in  $\tilde{S}(W)$ .

(iii)  $N \geq 0$  and  $N \geq M$ . We will show this in the remaining of the proof.

It is easy to see that formula (2.18) holds for  $\tilde{S}(W)$ . We will use formula (2.18) to simplify (6.1) and show that it is actually an element of  $O_g(V)$  acting on  $w \in W$  and hence equals 0 in  $\tilde{S}(W)$ . Note that

$$\text{Res}_y(1+y)^{\text{wt } u + N + \delta_r + \frac{r}{T}} y^m Y(u, y)v = \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} u_{j+m}v.$$

For simplicity, we say an element is of form  $O_N$  if it can be written as a linear combination of elements of the form:

$$\sum_{j=0}^{\infty} \binom{\text{wt } \tilde{u} + N + \delta_{\tilde{r}} + \frac{\tilde{r}}{T}}{j} (\tilde{u}_{j+n}\tilde{v})(\text{wt } \tilde{u}_{j+n}\tilde{v} + N - 1)\tilde{w}$$

with  $n \leq -N - \delta_{\tilde{r}} - \frac{\tilde{r}}{T} - 2$  for some  $\tilde{u} \in V^{\tilde{r}}$ ,  $\tilde{v} \in V$ .

Applying formula (2.18) to

$$\begin{aligned}
& a(\text{wt } a - M + N - 1) \\
& \cdot \left( \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) \tilde{w},
\end{aligned} \tag{6.2}$$

where  $\tilde{w} = b(\text{wt } b - N - 1)w \in \tilde{S}(W)$ , by specializing

$$\begin{aligned} p &= \text{wt } a - M + N - 1 \\ q &= \text{wt } u + \text{wt } v - j - m + M - 2 \\ k &= \text{wt } u + \text{wt } v - j - m + N - 1 \\ l &= \text{wt } a + N + \delta_d + \frac{d}{T}, \end{aligned}$$

we have

$$\begin{aligned} & a(\text{wt } a - M + N - 1) \\ & \cdot \left( \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) \tilde{w} \\ &= \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{N-M} x_0^{-M-i-\delta_d-\frac{d}{T}-1} \\ & \quad \cdot (x_2 + x_0)^{\text{wt } a+N+\delta_d+\frac{d}{T}} x_2^{\text{wt } u+\text{wt } v-j-m+M-2+i} Y_{\tilde{S}(W)}(Y(a, x_0)(u_{j+m}v), x_2) \tilde{w} \\ &= \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{N-M} (1+y)^{\text{wt } u+N+\delta_r+\frac{r}{T}} y^m x_0^{-M-i-\delta_d-\frac{d}{T}-1} \\ & \quad \cdot (x_2 + x_0)^{\text{wt } a+N+\delta_d+\frac{d}{T}} Y_{\tilde{S}(W)}(Y(a, x_0)x_2^{L(0)+M-1+i}Y(u, y)v, x_2) \tilde{w} \\ &= \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \sum_{i=0}^{N-M} (1+y)^{\text{wt } u+N+\delta_r+\frac{r}{T}} y^m x_0^{-M-i-\delta_d-\frac{d}{T}-1} x_2^{\text{wt } u+\text{wt } v+M-1+i} \\ & \quad \cdot (x_2 + x_0)^{\text{wt } a+N+\delta_d+\frac{d}{T}} Y_{\tilde{S}(W)}(Y(u, x_2y)Y(a, x_0)v, x_2) \tilde{w} \\ & \quad + \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{N-M} (1+y)^{\text{wt } u+N+\delta_r+\frac{r}{T}} y^m x_0^{-M-i-\delta_d-\frac{d}{T}-1} x_2^{\text{wt } u+\text{wt } v+M-1+i} \\ & \quad \cdot (x_2 + x_0)^{\text{wt } a+N+\delta_d+\frac{d}{T}} x_0^{-1} \delta\left(\frac{x_2y + x_1}{x_0}\right) Y_{\tilde{S}(W)}(Y(Y(a, x_1)u, x_2y)v, x_2) \tilde{w}. \end{aligned}$$

By examining the monomials in  $y$  in the first term of the right-hand side, we know that the first term of the right-hand side is a sum of elements of form  $O_N$ . We only need to prove

the second term is also a sum of elements of form  $O_N$ . The second term equals

$$\begin{aligned}
& \text{Res}_y \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{N-M} (1+y)^{\text{wt } u+N+\delta_r+\frac{r}{T}} y^m x_0^{-M-i-\delta_d-\frac{d}{T}-1} x_2^{\text{wt } u+\text{wt } v+M-1+i} \\
& \quad \cdot (x_2+x_0)^{\text{wt } a+N+\delta_d+\frac{d}{T}} x_0^{-1} \delta\left(\frac{x_2 y+x_1}{x_0}\right) Y_{\tilde{S}(W)}(Y(Y(a, x_1)u, x_2 y)v, x_2) \tilde{w} \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^{N-M} (1+y)^{\text{wt } u+N+\delta_r+\frac{r}{T}} y^m (x_2 y+x_1)^{-M-i-\delta_d-\frac{d}{T}-1} x_2^{\text{wt } u+\text{wt } v+M-1+i} \\
& \quad \cdot (x_2+x_2 y+x_1)^{\text{wt } a+N+\delta_d+\frac{d}{T}} Y_{\tilde{S}(W)}(Y(Y(a, x_1)u, x_2 y)v, x_2) \tilde{w} \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^{N-M} (1+y)^{\text{wt } u+\text{wt } a+2N+\delta_r+\delta_d+\frac{r+d}{T}+1} y^m (x_2 y+x_3(1+y))^{-M-i-\delta_d-\frac{d}{T}-1} \\
& \quad \cdot x_2^{\text{wt } u+\text{wt } v+M-1+i} (x_2+x_3)^{\text{wt } a+N+\delta_d+\frac{d}{T}} Y_{\tilde{S}(W)}(Y(Y(a, (1+y)x_3)u, x_2 y)v, x_2) \tilde{w} \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^{N-M} \sum_{j=0}^{\infty} \binom{-M-i-\delta_d-\frac{d}{T}-1}{j} (1+y)^{\text{wt } u+\text{wt } a+2N+\delta_r+\delta_d+\frac{r+d}{T}+1+j} \\
& \quad \cdot x_3^j y^{m-M-i-\delta_d-\frac{d}{T}-1-j} x_2^{\text{wt } u+\text{wt } v-\delta_d-\frac{d}{T}-2-j} (x_2+x_3)^{\text{wt } a+N+\delta_d+\frac{d}{T}} \\
& \quad \cdot Y_{\tilde{S}(W)}(Y(Y(a, (1+y)x_3)u, x_2 y)v, x_2) \tilde{w} \\
= & \text{Res}_y \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^{N-M} \sum_{j=0}^{\infty} \binom{-M-i-\delta_d-\frac{d}{T}-1}{j} \\
& \quad \cdot x_3^j x_2^{\text{wt } u+\text{wt } v-\delta_d-\frac{d}{T}-2-j} (x_2+x_3)^{\text{wt } a+N+\delta_d+\frac{d}{T}} \\
& \quad \cdot Y_{\tilde{S}(W)}(y^{m-M-i-\delta_d-\frac{d}{T}-1-j} Y((1+y)^{L(0)+2N+\delta_r+\delta_d+\frac{r+d}{T}+1+j} Y(a, x_3)u, x_2 y)v, x_2) \tilde{w}.
\end{aligned}$$

By checking the monomial in  $y$  of the right-hand side, it is the sum of elements of the form

$$\text{Res}_y y^{m'} Y(1+y)^{L(0)+2N+\delta_r+\delta_d+\frac{r+d}{T}+1+j} (Y(a, x_3)u, y)v, \quad (6.3)$$

where  $m' \leq m - M - i - \delta_d - \frac{d}{T} - 1 - j \leq -2N - \delta_r - \delta_d - \frac{r+d}{T} - 3 - j$ . Note that

$$2N + \delta_r + \delta_d + \frac{r+d}{T} + 1 + j \geq N + \delta_{\overline{r+d}} + \frac{\overline{r+d}}{T},$$

here we use  $\overline{r+d}$  to denote the residue of  $r+d$  modulo  $T$ , the element (6.3) can be written as an element of the form

$$\text{Res}_y y^{m''} Y(1+y)^{L(0)+N+\delta_{\overline{r+d}}+\frac{\overline{r+d}}{T}} (Y(a, x_3)u, y)v,$$

where  $m'' \leq m' + 2N + \delta_r + \delta_d + \frac{r+d}{T} + 1 + j - (N + \delta_{\overline{r+d}} + \frac{\overline{r+d}}{T}) \leq -N - \delta_{\overline{r+d}} - \frac{\overline{r+d}}{T} - 2$ .

Thus the element (6.3) is of form  $O_N$ . We proved that

$$a(\text{wt } a - M + N - 1) \cdot \left( \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} (u_{j+m}v)(\text{wt } u + \text{wt } v - j - m + M - 2) \right) \tilde{w}$$

is a sum of elements of the form

$$\sum_{j=0}^{\infty} \binom{\text{wt } \tilde{u} + N + \delta_{\tilde{r}} + \frac{\tilde{r}}{T}}{j} (\tilde{u}_{j+n}\tilde{v})(\text{wt } \tilde{u}_{j+n}\tilde{v} + N - 1)\tilde{w}$$

with  $n \leq -N - \delta_{\tilde{r}} - \frac{\tilde{r}}{T} - 2$  for some  $\tilde{u} \in V^{\tilde{r}}$ ,  $\tilde{v} \in V$ . For simplicity, we still write this element as

$$\sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} (u_{j+m}v)(\text{wt } u_{j+m}v + N - 1)b(\text{wt } b - N - 1)w \quad (6.4)$$

for  $m \leq -N - \delta_r - \frac{r}{T} - 2$ ,  $u \in V^r$  and  $w \in W$ .

We shall prove that (6.4) is 0 in  $\tilde{S}(W)$ . Applying formula (2.18) to the expression (6.4), by specializing

$$\begin{aligned} p &= \text{wt } u_{j+m}v + N - 1 \\ q &= \text{wt } b - N - 1 \\ k &= \text{wt } b \\ l &= \text{wt } u_{j+m}v + \delta_{r+s} + \frac{\overline{r+s}}{T}, \end{aligned}$$

here we use  $\overline{r+s}$  to denote the residue of  $r+s$  modulo  $T$ , we have

$$\begin{aligned}
& \sum_{j=0}^{\infty} \binom{\text{wt } u + N + \delta_r + \frac{r}{T}}{j} (u_{j+m}v) (\text{wt } u_{j+m}v + N - 1) b (\text{wt } b - N - 1) w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_y \sum_{i=0}^N x_0^{N-i-\delta_{r+s}-\frac{\overline{r+s}}{T}-1} x_2^{\text{wt } b-N-1+i} (1+y)^{\text{wt } u+N+\delta_r+\frac{r}{T}} y^m \\
&\quad \cdot Y_{\tilde{S}(W)}(Y((x_2+x_0)^{L(0)+\delta_{r+s}+\frac{\overline{r+s}}{T}} Y(u,y)v, x_0) b, x_2) w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_y \sum_{i=0}^N x_0^{N-i-\delta_{r+s}-\frac{\overline{r+s}}{T}-1} x_2^{\text{wt } b-N-1+i} (1+y)^{\text{wt } u+N+\delta_r+\frac{r}{T}} y^m \\
&\quad \cdot (x_2+x_0)^{\text{wt } u+\text{wt } v+\delta_{r+s}+\frac{\overline{r+s}}{T}} Y_{\tilde{S}(W)}(Y(Y(u, (x_2+x_0)y)v, x_0) b, x_2) w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_3} \sum_{i=0}^N x_0^{N-i-\delta_{r+s}-\frac{\overline{r+s}}{T}-1} x_2^{\text{wt } b-N-1+i} (x_2+x_0+x_3)^{\text{wt } u+N+\delta_r+\frac{r}{T}} x_3^m \\
&\quad \cdot (x_2+x_0)^{\text{wt } v-N+\delta_{r+s}+\frac{\overline{r+s}}{T}-\delta_r-\frac{r}{T}-m-1} Y_{\tilde{S}(W)}(Y(Y(u, x_3)v, x_0) b, x_2) w \\
&= \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^N x_0^{N-i-\delta_{r+s}-\frac{\overline{r+s}}{T}-1} x_2^{\text{wt } b-N-1+i} (x_2+x_1)^{\text{wt } u+N+\delta_r+\frac{r}{T}} (x_1-x_0)^m \\
&\quad \cdot (x_2+x_0)^{\text{wt } v-N+\delta_{r+s}+\frac{\overline{r+s}}{T}-\delta_r-\frac{r}{T}-m-1} Y_{\tilde{S}(W)}(Y(u, x_1) Y(v, x_0) b, x_2) w \\
&\quad - \text{Res}_{x_0} \text{Res}_{x_2} \text{Res}_{x_1} \sum_{i=0}^N x_0^{N-i-\delta_{r+s}-\frac{\overline{r+s}}{T}-1} x_2^{\text{wt } b-N-1+i} (x_2+x_1)^{\text{wt } u+N+\delta_r+\frac{r}{T}} (-x_0+x_1)^m \\
&\quad \cdot (x_2+x_0)^{\text{wt } v-N+\delta_{r+s}+\frac{\overline{r+s}}{T}-\delta_r-\frac{r}{T}-m-1} Y_{\tilde{S}(W)}(Y(v, x_0) Y(u, x_1) b, x_2) w
\end{aligned}$$

By checking the monomials involving  $x_1$  in the first term of the right-hand side of the equation above, the first term is an action of elements in  $O_g(V)$  on  $w$ , that is 0 in  $\tilde{S}(W)$ .

For the second term, we shall check the monomials involving  $x_0$ . It is the action of an element of the form

$$\text{Res}_{x_0} (1+x_0)^{\text{wt } v-N+\delta_{r+s}+\frac{\overline{r+s}}{T}-\delta_r-\frac{r}{T}-m-1} x_0^{m'} Y(v, x_0) \tilde{u} \quad (6.5)$$

on  $W$ , where  $m' \leq m + N - i - \delta_{r+s} - \frac{\overline{r+s}}{T} - 1$ . Note that

$$-N + \delta_{r+s} + \frac{\overline{r+s}}{T} - \delta_r - \frac{r}{T} - m - 1 \geq \delta_s + \frac{s}{T} - 1,$$

the element (6.5) can be written as an element of the form

$$\text{Res}_{x_0} (1+x_0)^{\text{wt } v+\delta_s+\frac{s}{T}-1} x_0^{m''} Y(v, x_0) \tilde{u},$$

where  $m'' = m' - N + \delta_{\overline{r+s}} + \frac{\overline{r+s}}{T} - \delta_r - \frac{r}{T} - m - 1 - \delta_s - \frac{s}{T} + 1 \leq -1 - \delta_s$ , hence the element (6.5) lies in  $O_g(V)$  and the second term is 0 in  $\tilde{S}(W)$ . ■

Let  $S_g(W) = \tilde{S}(W)/\mathcal{J}$ . Then  $S_g(W)$  is also a  $\frac{1}{T}\mathbb{N}$ -graded  $T(V^g[t, t^{-1}])$ -module. We can still use elements of  $T(V^g[t, t^{-1}]) \otimes W$  to represent elements of  $S_g(W)$ . But note that these elements now satisfy relations. We equip  $S_g(W)$  with the vertex operator map

$$Y_{S_g(W)} : V \otimes S_g(W) \longrightarrow S_g(W)[[x^{\frac{1}{T}}, x^{-\frac{1}{T}}]]$$

given by

$$u \otimes w \rightarrow Y(u, x)w = Y_t(u, x)w.$$

**Theorem 6.2** *The pair  $(S_g(W), Y_{S_g(W)})$  is an admissible  $g$ -twisted  $V$ -module such that  $(S_g(W))_0 = W$ .*

*Proof.* As in  $\tilde{S}(W)$ , for  $u \in V$  and  $w \in S_g(W)$ , we also have  $u(m)w = 0$  when  $m > \text{wt } u + \deg w - 1$ . Clearly,

$$Y_{S_g(W)}(\mathbf{1}, x) = I_{S_g(W)},$$

where  $I_{S_g(W)}$  is the identity operator on  $S_g(W)$ .

By Theorem 2.1, where we specialize  $k = \text{wt } v + \deg w$  and  $l = \text{wt } u + \deg w + \delta_r + \frac{r}{T}$  for  $u \in V^r$ ,  $S_g(W)$  satisfies weak associativity and hence is an admissible  $g$ -twisted  $V$ -module by Theorem 2.5. The claim that  $(S_g(W))_0 = W$  follows from Lemma 6.1. ■

**Theorem 6.3** *The functor  $S_g$  has the following universal property: Let  $W$  be an  $A_g(V)$ -module. For any admissible twisted  $V$ -module  $\tilde{W}$  and any  $A_g(V)$ -module map  $f : W \rightarrow \Omega(\tilde{W})$ , there exists a unique  $V$ -module map  $\tilde{f} : S_g(W) \rightarrow \tilde{W}$  such that  $\tilde{f}|_W = f$ .*

In [DLM2], a functor, denoted by  $\bar{M}$ , was constructed explicitly and was proved to satisfy the same universal property above. The following result achieves our goal of constructing this functor without dividing relations corresponding to the commutator formula for weak modules:

**Corollary 6.4** *The functor  $S_g$  is equivalent to the functor  $\bar{M}$  constructed in [DLM2].*

*Proof.* This result follows immediately from the universal property. ■

## REFERENCES

- [DLM1] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Alg.* **206** (1998), 67–98.



- [DLM2] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571–600.
- [DLM3] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras and associative algebras, *Internat. Math. Res. Notices.* **8** (1998), 389–397.
- [H] Y.-Z. Huang, Generalized twisted modules associated to general automorphisms of a vertex operator algebra, *Comm. Math. Phys.* **298** (2010), 265–292.
- [HY1] Y.-Z. Huang and J. Yang, On functors between module categories for associative algebras and for  $\mathbb{N}$ -graded vertex algebras, *J. Alg.* **409** (2014), 344–361.
- [HY2] Y.-Z. Huang and J. Yang, Logarithmic intertwining operators and associative algebras, *J. Pure Appl. Alg.* **216** (2012), 1467–1492.
- [L] H. Li, The regular representation, Zhu  $A(V)$ -theory and induced modules, *J. Alg.* **238** (2001), 159–193.
- [LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math., Vol. 227, Birkhäuser, Boston, 2003.
- [MT] M. Miyamoto and K. Tanabe, Uniform product of  $A_{g,n}(V)$  for an orbifold model  $V$  and  $G$ -twisted Zhu algebra, *J. Alg.* **274** (2004), 80–96.
- [VE] J. Van Ekeren, Higher level twisted Zhu algebras, *J. Math. Phys.* **52**, 052302 (2011).
- [Z] Y.-C. Zhu, Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.* **9** (1996), 237–302.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, 278 HURLEY BUILDING, NOTRE DAME, IN 46556

*E-mail address:* jyang7@nd.edu